The S–D mixing and di-electron widths of higher charmonium 1^{--} states

A.M. Badalian,

B.L.G. Bakker (VU, Amsterdam),

I.V. Danilkin (ITEP)

Institute of Theoretical and Experimental Physics, Moscow, Russia

PANDA Meeting, 22 October, ITEP

Contents

1. The di-electron width as a test of the nature of a resonance.
2. Exp.data as the arguments in favor of the $S-D$ mixing of higher $c\bar{c}$ levels.
3. Shortly about the $c\bar{c}$ spectrum: the masses of the nS, nD states with $n = 1, 2, ..., 7$.
4. Di-electron widths of physical resonances: $\psi(4040)$ and $\psi(4160), \psi(4420)$ and $\psi(4360)$, $\psi(4650)$ and $\psi(4690)$, $\psi(4820), \psi(4950)$.

1. The di-electron width as an important test of the nature of a resonance.

Knowledge of the di-electron widths of higher charmonium states is important for many reasons.

It can help to distinguish between conventional $c\bar{c}$ mesons and, for example, tetraquarks, which have much smaller di-electron widths than four-quark systems, e.g. $c\bar{c}q\bar{q}$, with $J^{PC} = 1^{--}$ have Γ_{ee} by ~ 100 times smaller than $\Gamma_{ee}(c\bar{c})$.

This effect occurs because the probability to collect four (and even three) particles at the origin is much smaller than for two particles. (This fact does not exclude that at larger distances the continuum part can give an essential contribution to the w.f. and even dominates asymptotically). Therefore the w.f. at the origin of a higher vector resonance can be calculated taking into account the mixing between neighbouring vector states.

Study of the charmonium spectrum shows that the $(n+1)\,^3S_1$ and $n\,^3D_1$ states $(n \geq 3)$ have small mass differences, ≤ 50 MeV, which decrease for higher radial excitations.

On the other hand, the mass differences between neighbouring $n\ ^3S_1$ states are of the order several hundreds of MeV, so in first approximation mixing between these states can be neglected.

The S–D mixing between $\psi'(3686)$ and $\psi''(3770)$ is not large: $\theta = (12 \pm 2)^\circ$ has been extracted from the ratio of their di-electron widths.

We show that for higher vector states the S–D mixing is significantly larger and the mixing angle $\theta \sim 34^\circ$.

2. Experimental facts as the arguments in favor of the $S - D$ mixing

Meson	dominant state	$\Gamma_{ee}(exp)$ (keV):
J/ψ	$1\,^3S_1$	5.55 ± 0.16
$\psi\,'$	$2\,^3S_1$	2.48 ± 0.06
$\psi(3770)$	1^3D_1	0.24 ± 0.03
$\psi(4040)$	$3\,^3S_1$	0.86 ± 0.07
$\psi(4159)$	2^3D_1	0.83 ± 0.07
$\psi(4415)$	$4\,^3S_1$	0.58 ± 0.07

In the recent paper of DZY the $Y(4660)$ resonance is considered as a $5\,^3S_1$ state with large $\Gamma_{ee}(5\,^3S_1) = 1.34$ keV, which is even significantly larger than $\Gamma_{ee}(\psi(4415)) = 0.58(7)$ keV.

1. The measured di-electron width of $\psi(4160)$, which is usually considered as the $2\,^3D_1$ state, is large,

$$\Gamma_{ee}(\psi(4160)) = 0.83 \pm 0.06 \text{ keV},$$

It is only 5–10% smaller than $\Gamma_{ee}$of $\psi(4040)$ and $\sim$14 times larger than the width for a pure $2\,^3D_1$ state: $\Gamma_{ee}(2\,^3D_1) = 0.061$ keV. It is also even three times larger than $\Gamma_{ee}(\psi''(3770)) = 0.24(3)$ keV.

2. On the contrary,

$$\Gamma_{ee}(\psi(4040))|_{exp} = 0.86 \pm 0.07 \text{ keV},$$

appears to be almost two times smaller than for a pure $3\,^3S_1$ state. This situation can be resolved if the S–D mixing between these two states is taken into account. This mixing can occur due to an influence of open channel, while a mixing owing to the short-range tensor forces gives a rather small effect.

3. The di-electron width of $\psi(4415)$. If this resonance is considered as the $4\,^3S_1$ state, then the best potential models calculations give $\Gamma_{ee} \sim 1.1 - 1.5$ keV which is two-three times larger than the experimental number,

$$\Gamma_{ee}(\psi(4415)) = 0.58 \pm 0.07 \text{ keV}.$$

Again such a decrease of the di-electron width could occur via $4S - 3D$ mixing. In our calculations without hadronic shifts $M(3D) = 4.470(20)$ MeV, while $M(4S) = 4410(20)$ MeV, i.e., these two masses are rather close to the masses of the physical resonances $\psi(4415)$ and $Y(4360)$. These $4\,^3S_1$ and $3\,^3D_1$ states could be strongly coupled to the S-wave decay channels, like $D_1(2420)D^*(2010)$, $D_0^*(2400)D^*(2010)$, and $D_{s0}^*(2317)$ $D_s^*(2112)$, and due to this coupling the $4\,^3S_1$ and $3\,^3D_1$ levels aquire hadronic downward mass shifts, which are typically $\sim$ 40–60

MeV (PW paper).

4. It is convenient to define the mixing angle between higher vector states from the ratio of the di-electron widths, as for $\psi(3686)$ and $\psi(3770)$ when not well known factors, like the QCD radiative corrections, are cancelled.

5. The mass of the $5\,^3S_1$ state, $M(5\,^3S_1) = 4650(20)$ MeV, was predicted in BB paper, before the Belle resonance $Y(4660)$ was discovered. If this resonance is apure $5\,^3S_1$ state DZY give $\Gamma_{ee}(Y(4660)) = 1.34$ keV; in our calculations $\Gamma(Y(4660)) = 0.70$ keV, due to our choice of potential. And about two times smaller, $\Gamma_{ee}(Y(4660)) = 0.31$ keV, if the mixing angle between $5\,^3S_1$ and the unobserved $4\,^3D_1$ state (with mass ~ 4700 MeV) is 34°, (the same as for $\psi(4415)$ and $\tilde{\psi}(3D)$).

3. The masses of the $J^{PC} = 1^{--}$ charmonium states

The hyperfine (HF) and fine-structure splittings of higher radial excitations are small (≤ 20 MeV), therefore their masses practically coincide with the centroid masses, $M_{\rm cog}(nL)$. To calculate them we use the relativistic string Hamiltonian (RSH) with universal (for all mesons) interaction.

For RSH in heavy quarkonia the mass $M_{\rm cog}(nL)$ is just given by the e.v. of the spinless Salpeter equation (SSE):

$$\left\{2\sqrt{\mathbf{p}^2 + m_c^2} + V_B(r)\right\} \psi_{nL}(r) = M_{\rm cog}(nL)\psi_{nL}(r).$$

We also use the RS Hamiltonian, written in einbein approximation (EA), when the spin-averaged mass is presented as:

$$M_{\rm cog}(nL) = \omega_{nL} + \frac{m_c^2}{\omega_{nL}} + E_{nL}(\omega_c),$$

The self-energy correction $\Delta_{SE}(c\bar{c}) \cong -20$ MeV can be included in the definition of the pole of c quark.

The e.v. E_{nL} are the solutions of the so-called einbein equation:

$$\left[\frac{\vec{p}^2}{\omega_{nL}} + V_B(r)\right] \varphi_{nL}(r) = E_{nL}\varphi_{nL},$$

defined together with the dynamical mass ω_{nL} in a selfconsistent way:

$$\omega_{nL}^2 = m_c^2 - \frac{\partial E_{nL}}{\partial \omega_{nL}}.$$

For the $n\,^3D_1$ states we take $M(n\,^3D_1) = M_{\rm cog}(nD)$, while for the $n\,^3S_1$ states $M(n\,^3S_1) = M_{\rm cog}(nS) + \frac{1}{4}\delta_{\rm HF}(nS)$ with $\delta_{\rm HF}(nS) = M(n\,^3S_1) - M(n^1S_0)$. The values of $\delta_{\rm HF}(nS) = 48(48)$, 16(20), 12(16), 6(10) MeV $(n = 2, 3, 4, 5)$, are calculated in BGS, BB papers.

The universal potential $V_B(r)$ is taken from the background perturbation theory:

$$V_B(r) = \sigma(r)\cdot r - \frac{4}{3}\frac{\alpha_B(r)}{r},$$

where the vector coupling $\alpha_B(r)$ it has the asymptotic freedom behavior at small r, freezes (saturates) at large r, and depends on the number of flavors n_f (for charmonium $n_f = 4$).

For low-lying states with r.m.s $R(nL) \leq 0.8$ fm a linear confining potential with constant string tension, $\sigma = \sigma_0 \cong 0.18$ GeV^2, is used. However, for higher states, which lie above open thresholds and have sizes $R(nL) \geq 1.0$ fm, it is important to take into account the creation of virtual light-quark pairs $(q\bar{q})$. It can be done even in single-channel approximation. Due to virtual loops the surface inside the Wilson loop decreases, making the string tension dependent on

$$\sigma(r) = \sigma_0(1 - \gamma f(r)), \quad \gamma = 0.40, \quad \sigma_0 = 0.18 \text{ GeV}^2$$

$$f(r \to 0) = 0, \quad f(r \to \infty) = 1.0.$$

Such a flattening of the confining potential is common to all mesons of large sizes and therefore for charmoium the parameters of $\sigma(r)$ can be taken from the analysis of the radial Regge trajectories for light mesons (BBS, 2002).

Table 1. The charmonium masses $M(n\,^3S_1)$ (in MeV) ($m_c = 1.42$ GeV, $\sigma_0 = 0.18$ GeV^2)

state	SSE $\sigma(r)$	EA $\sigma(r)$	EA, $\sigma_0 = const$	exp.
1S	3100	3095	3100	3097
2S	3686	3682	3690	3686
3S	4075	4096	4116	4039(1)
4S	4398	4426	4470	4421(4) 4361(18)
5S	4642	4672	4784	4664(16)
6S	4804	4828	5070	
7S	4950	4980	5375	

Table 2. The charmonium masses $M(n\,^3D_1)$ (in MeV) ($m_c = 1.42$ GeV, $\sigma_0 = 0.18$ GeV^2

state	SSE, $\sigma(r)$	EA$\sigma(r)$	EA, $\sigma_0 = const$	Exp.
1D	3800	3779	3789	3770(3) PDG
2D	4165	4165	4288	4159(3)PDG
3D	4465	4477	4523	4421 PDG 4361, 4324 Belle, BaBar
4D	4690	4707	4825	4664(16) Belle 4634(14) Belle
5D	4840	4855	5100	
6D	4975	5005	5407	

Our analysis shows that the mass difference,

$$\Delta_n M = M(n\,^3D_1) - M((n+1)\,^3S_1),$$

decreases from the value $\Delta_2 M(\exp) = 120$ MeV for $n = 2$ up to ~ 30 MeV for $n = 4$. Therefore higher levels are almost degenerate and for them the S–D mixing is probable. Also in single-channel approximation the $M_{\rm cog}(n+1)S$ $M_{\rm cog}(nD)$. Could this order be changed due to hadronic shifts?

However, our mass of the $3\,^3S_1$ level is ~ 40 MeV larger than the experimental one, because this level can be affected by open $D^*\bar{D}^*$ channel and DC shift ~ 40 MeV is usually estimated.

4. Di-electron widths

The di-electron width is expressed via vector decay constant, containing the relativistic correction ξ_V, and includes QCD radiative corrections, known in one-loop approximation, which enters as the multiplicative factor denoted here as $\beta_V = 1 - \frac{16}{3\pi}\alpha_s(M_V)$: Then

$$\Gamma_{ee}(n\,^3S_1) = \frac{4\pi e_c^2\alpha^2}{3M_{nS}} f_{nS}^2\beta = \frac{4e_c^2\alpha^2}{M_{nS}^2}|R_{nS}(0)|^2\xi_{nS}\beta_V,$$

$$\Gamma_{ee}(n\,^3D_1) = \frac{4\pi e_c^2\alpha^2}{3M_{nD}} f_{nD}^2\beta = \frac{4e_c^2\alpha^2}{M_{nD}^2}|R_{nD}(0)|^2\xi_{nD}\beta_V.$$

The w.f. at the origin $R_{nD}(0)$ is defined

$$R_{nD}(0) = \frac{5}{2\sqrt{2}\omega_{nD}^2} R''_{nD}(0).$$

and the average kinetic energy $\omega_{nL} = \langle\sqrt{\mathbf{p}^2 + m_c^2}\rangle_{nL}$, plays the role

of a constituent quark mass, different for different nL states.
$\omega_{nL} \sim 1.65$ GeV for $n \geq 5$.

The w.f. at the origin $R_{nS}(0)$, $R_{nD}(0)$, and $R''_{nD}(0)$ are of pure nS and nD states are calculated with the use of RSH. The w.f. of the physical ψ mesons are denoted as $\varphi_{nS}(0)$ and $\varphi_{nD}(0)$, where the symbols nS and nD simply remind about the origin of those states:

$$\begin{aligned} \varphi_{nS}(0) &= \cos\theta_n R_{nS}(0) - \sin\theta_n R_{(n-1)D}(0), \\ \varphi_{nD}(0) &= \cos\theta_n R_{(n+1)S}(0) - \sin\theta_n R_{nD}(0). \end{aligned} \tag{1}$$

The mixing angle is a fitting parameter

Table 3. The wave functions at the origin $\varphi_{(n+1)S}(0)$ and $\varphi_{nD}(0)$ (in $\mathrm{GeV}^{3/2}$) of the physical states for $n = 1, 2, 3, 4$ [a)].

n	1	2	3	4
θ	11°	34.8°	34°	34°
$\varphi_{(n+1)S}(0)$	0.735	0.511	0.459	0.360
$\varphi_{nD}(0)$	0.240	0.516	0.491	0.416

The vector decay constants, calculated without $S - D$ mixing ($\theta = 0$) and with $\theta \neq 0$ are given in Table 4.

Table 4. The decay constants $f_V(n\,^3S_1)$ (in MeV), calculated without and with S–D mixing.

$\theta = 0$			
	$n=1$	$n=2$	$n=3$
$f_V((n+1)\,^3S_1)$	373	329	288
$f_V(n\,^3D_1)$	45	60	66
$\theta \neq 0$			
θ	11°	34.8°	34°
$f_V(\psi_S)$	357	236	202
$f_V(\psi_D)$	115	234	217

For large S–D mixing all decay constants $\sim 220 \pm 20$ MeV and have close values of di-electron widths of the $(n+1)S$ and nD states.

A. $3\,^3S_1$–$2\,^3D_1$ mixing

The mixing angle between the $(n+1)\ ^3S_1$ and $n\ ^3D_1$ states, denoted here as θ_n, can be calculated if at least one of the di-electron widths is known from experiment. For the $3S$ and $2D$ states both di-electron widths are known and θ_2 is easily determined.

Notice that for a pure $2\,^3D_1$ state the di-electron width is very small: $\Gamma_{ee}(2\,^3D_1)$=0.059 keV, i.e., ~ 14 times smaller than the experimental number for $\psi(4160)$ and one can expect large mixing between the $3\ ^3S_1$ and $2\ ^3D_1$ states. Such a large mixing can occur via the nearby open $D^*\bar{D}^*$ channel and partly through short-ranged tensor forces, which, however, do not provide a large mixing angle, θ(tensor)$\lesssim 7°$. From the ratio:

$$\eta = \frac{\Gamma_{ee}(\psi(4040))}{\Gamma_{ee}(\psi(4160))} = 1.04 \pm 0.17$$

one obtains two solutions with a large magnitude of θ_2: a positive and a negative one:

$$\theta_2 = 34.8^\circ \quad \text{or} \quad \theta_2 = -55.7^\circ.$$

For these angles from (1) the physical w.f. $\varphi(\psi(4040), r = 0) = 0.511$ GeV$^{3/2}$ and $\varphi(\psi(4160), r = 0) = 0.516$ GeV$^{3/2}$ appear to be almost equal. Then from (3) and (3) with $\beta_V = 0.63$ one calculates following di-electron widths:

$$\Gamma_{ee}(\psi(4040))|_{th} = 0.87 \text{ keV}, \quad \Gamma_{ee}(\psi(4160))|_{th} = 0.83 \text{ keV},$$

$$\Gamma_{ee}(exp) = 0.86 \pm 0.07 \text{ keV}, \quad \Gamma_{ee}((exp) = 0.83 \pm 0.06 \text{ keV},$$

which just coincide with the central values of experimental numbers. The QCD factor $\beta = 0.63$, extracted from the absolute value, corresponds to the strong coupling $\alpha_s(M_V) = 0.217$. This value of $\beta_V = 0.63$, is also used for higher excitations $(n = 3, 4)$.

B. Large mixing between $4\,^3S_1$ and $3\,^3D_1$ states

In PM two vector states, $4\,^3S_1$ and $3\,^3D_1$, are expected in the mass region around 4.4 GeV. Our calculations give $M(4\,^3S_1) \sim 4.42$ GeV and $M(3\,^3D_1) \sim 4.47(1)$ GeV with $\Delta(M) \sim 50$ MeV. Due to strong coupling to the $D^*D_1(2420)$ and $D^*D_2^*(2460)$ channels the $4\,^3S_1$ and $3\,^3D_1$ states can be mixed and the mass of one or probably both states is shifted down: then one of these mixed (physical) states can be identified with $\psi(4415)$ and the other one with the Belle resonance $Y(4360)$.

At present from experiment only the $\Gamma_{ee}(\psi(4415)) = 0.58 \pm 0.07$ keV, is known, while for the $Y(4360)$ resonance two possible numbers are measured for the following product:

$$B(Y(4360) \to \psi(2S)\pi^+\pi^-)\;\Gamma_{ee}(Y(4360)) = \begin{array}{ll} a) & 10.4 \pm 3.2 \text{ eV}, \\ b) & 11.8 \pm 3.2 \text{ eV}. \end{array}$$

For pure $4S$ and $3D$ states ($\theta_3 = 0$) with the w.f. at the origin $R_{4S}(0) = 0.655$ GeV$^{3/2}$ and $R_{3D}(0) = 0.150$ GeV$^{3/2}$:

$$\Gamma_{ee}(4\,^3S_1) = 1.19 \text{ keV}, \quad \Gamma_{ee}(3\,^3D_1) = 0.06 \text{ keV}, \quad (\theta = 0).$$

while $\Gamma_{ee}(\psi(4415))|_{exp} = 0.58 \pm 0.07$ keV i.e., $\Gamma_{ee}(4S)$ is two times larger than the experimental number and $\Gamma_{ee}(3D)$ is very small. To reach agreement with experiment for $\psi(4415)$ one needs take a large mixing angle, $\theta_3 = 34^\circ$ (the same as for the $3S$–$2D$ mixing), then

$$\Gamma_{ee}(\psi(4415))|_{\text{theory}} = 0.57 \text{ keV}$$

in agreement with experimental value.

For the same $\theta = 34^o$ the di-electron width of second physical state, denoted as $\tilde{\psi}(4470)$ is:

$$\Gamma_{ee}(\tilde{\psi}(4470)) = 0.63 \text{ keV},$$

we take $\beta_V = 0.63$ as for the $3S$ and $2D$ states, i.e. the di-electron widths coincide within 10% accuracy, and in the single-channel approximation it is difficult to conclude which of these states should be identified with the $\psi(4415)$ meson or with the $Y(4360)$ resonance. From experimental product $B\Gamma_{ee}$ and the di-electron width one can obtain an estimate of the branching $B(Y(4360) \to \psi(2S)\pi^+\pi^-)$,

$$B(Y(4360) \to \psi(2S)\pi^+\pi^-) \approx (1.6 \pm 0.6)\%,$$

C. The $Y(4660), Y(4815)$

The Belle resonance $Y(4660)$ (2007) observed in $e^+e^- \to \psi(2S)\pi^+\pi^-$ with $M = 4660 \pm 16$ MeV, and recently discovered $Y(4634)$ in $e^+e^- \to \Lambda_c^+\Lambda_c^-$ with $M = 4634 + \binom{13}{15}$ MeV and $\Gamma = 92^{+500}_{-45}$ MeV can be considered as the same resonance. Its mass is close to the S-wave threshold $D_s^* D_{s1}(2535)$ ($M_{\text{th}} = 4647$ MeV) and the P-wave threshold $D(2\,^3S_1)\bar{D}^*$ with $M_{\text{th}} = 4647$ MeV (our calculations give $M(D(2\,^3S_1)) \approx 2640$ MeV). In single-channel approximation

$$M(5\,^3S_1) = 4655(15th)\text{ MeV}, \quad M(4\,^3D_1) = 4700(15th)\text{ MeV}.$$

These masses differ by only ~ 50 MeV and large $S - D$ mixing between two states can be expected. Unfortunately, at present their di-electron widths remain unknown, and we give their numbers with QCD factor $\beta_V = 0.63$ as for $4S$ and $3D$ states. For pure $5S$ and $4D$

states ($\theta_4 = 0$)

$$\Gamma_{ee}(5\,^3S_1) = 0.73 \quad \text{keV}, \quad \Gamma_{ee}(4\,^3D_1) = 0.055 \text{ keV } (\theta_4 = 0),$$

i.e., $\Gamma_{ee}(5S)$ would be larger than $\Gamma_{ee}(4S)$, on the contrary, $\Gamma_{ee}(4D)$ is very small.

For large S–D mixing with $\theta_4 = 34^\circ$ (as θ_2 and θ_3) we obtain two times smaller number:

$$\Gamma_{ee}(\tilde{\psi}(4660)) = 0.32 \text{ keV} \quad (\theta_4 = 34^\circ).$$

Then second physical state, denoted as $\tilde{\psi}(4690)$, has

$$\Gamma_{ee}(\tilde{\psi}(4690)) = 0.45 \text{ keV} \quad (\theta_4 = 34^\circ),$$

which is eight times larger than for the pure $4D$ state in (3) and even larger than $\Gamma_{ee}(\tilde{\psi}(4660))$. Equal widths Γ_{ee} are obtained for $\theta_4 = 30^\circ$:

$$\Gamma_{ee}(\tilde{\psi}(4660)) = \Gamma_{ee}(\tilde{\psi}(4690)) = 0.39 \text{ keV}.$$

D. The $6\,{}^3S_1$ state. It has very large r.m.s., $R \cong 2.5$ fm, even in closed-channel approximation. Still light mesons of such large size, e.g. $\rho(4S)$, exist. Its has $M(6\,{}^3S_1) = 4815 \pm 15$ MeV and $\Gamma_{ee}(6\,{}^3S_1) = 0.20$ keV for $\theta_5 = 34°$. The observation of so high a resonance would be important for the theory.

For characteristic angle $\theta = 30^0$ our predictions are

$$\Gamma_{ee}(5\ {}^3S_1) = 0.40\ keV, \qquad M(5^3S_1) = 4.66\ GeV$$

$$\Gamma_{ee}(4\ {}^3D_1) = 0.40\ keV, \qquad M(4^3D_1) = 4.69\ GeV$$

$$\Gamma_{ee}(6\ {}^3S_1) = 0.26\ keV, \qquad \Gamma_{ee}(5\ {}^3D_1) = 0.30\ keV\ !$$

$$M(6^3S_1) = 4.815\ GeV, \qquad M(5^3D_1) = 4.84\ GeV$$

There is no other way to have large Γ_{ee} for $\psi(2D)$ ($\Gamma_{ee} = 0.83 \pm 0.07$ keV), as with $\theta \approx 30^0$.

Conclusions

Our study of the di-electron widths of higher $n\,^3S_1$ and $n\,^3D_1$ radial excitations in charmonium shows that:

1. The almost equal values of $\Gamma_{ee}(4040)$ and $\Gamma_{ee}(4160)$, as well as the small value of $\Gamma_{ee}(4415)$, can be explained, if large S–D mixing between $(n+1)\,^3S_1$ and $n\,^3D_1$ states takes place.

2. For $\psi(4040)$ and $\psi(4160)$ precise agreement with experiment is obtained taking the mixing angle $\theta_2 = 34.8^\circ$.

3. For $\psi(4415)$ the calculated di-electron width coincides with the central value determined by experiment for a mixing angle $\theta_3 = 34^\circ$.

4. In all cases the QCD radiative corrections appear to be important and the same strong coupling $\alpha_s(M_V) = 0.217$ is taken, giving $\sim 30\%$ effect.

5. The DC (decay channel) mass shifts (due to strong coupling to a nearby threshold) are not calculated here . Therefore it remains unclear which physical resonance, $\psi(4415)$ or the Belle resonance $Y(4360)$, corresponds to the $3\,^3D_1(4\,^3S_1)$ state. For both states we predict close values of their di-electron widths: $\Gamma_{ee}(\psi(4415)) = 0.57$ keV and $\Gamma_{ee}(Y(4360)) \cong 0.63(7)$keV (they coincide within the experimental error).

6. Assuming that the $5\,^3S_1$ and $4\,^3D_1$ states have also large S–D mixing, with $\theta_n = (32 \pm 2)^\circ$), we obtain:$\Gamma_{ee}(\tilde{\psi}(4660)) = 0.35(4)$ keV, $\Gamma_{ee}(\tilde{\psi}(4690)) = 0.40(5)$ keV.

7. One cannot exclude that $6\,^3S_1$ state also exists, for which $M(6S) = 4815(15)$ MeV and the di-electron width $\Gamma_{ee} = 0.20$ keV are predicted.